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TWISTED SUMS AND A PROBLEM OF KLEE

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To Victor Klee

ABSTRACT

Let F be a quasi-linear map on a separable normed space E, and assume that F splits on an infinite-dimensional subspace of E. Then the twisted sum topology on $\mathbb{R}\otimes_F E$ can be written as the supremum of a nearly convex topology and a trivial dual topology. (This partially answers a question of Klee.) The result applies to the Ribe space and to James's space.

In [5], Klee asked whether every vector topology τ on a real vector space X is the supremum of a nearly convex topology τ_1 and a trivial dual topology τ_2. Recall that a vector topology τ_1 on X is **nearly convex** if for every x not in the τ_1-closure of $\{0\}$ there is f in $(X, \tau_1)^*$ with $f(x) \neq 0$; τ_2 is **trivial dual** if $(X, \tau_2)^* = \{0\}$. We do not require that τ_1 or τ_2 be Hausdorff, even if τ itself is Hausdorff. The topology τ is the **supremum** of τ_1 and τ_2 if τ_1 and τ_2 are weaker than τ, and if for every τ-neighborhood U of the origin 0 there are a τ_1-neighborhood V of 0 and a τ_2-neighborhood W of 0 such that $U \supset V \cap W$.

In [5], Klee proved that the usual topology on ℓ_p, $0 < p < 1$, is not the supremum of a locally convex topology and a trivial dual topology; this and other examples make the question at the beginning of this paper a natural one. Some related questions on suprema of linear topologies were studied in [7].

Given any vector topology τ on X, let $K(\tau) = \cap\{f^{-1}(0)\colon f \in (X,\tau)^*\}$. It is trivial to answer Klee's question affirmatively in the case that $K(\tau)$ is complemented. For in this case, $K(\tau)$ must be a trivial dual space in the relative topology; and if L is a complement to $K(\tau)$ in X, the relative topology on L is

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nearly convex. Now simply let τ_1 be the product of the trivial topology on $K(\tau)$ and the relative topology on L; and let τ_2 be the product of the relative topology on $K(\tau)$ and the trivial topology on L. Then $\tau = \sup(\tau_1, \tau_2)$.

So the interesting case is when $K(\tau)$ is uncomplemented. We study the problem when (X, τ) is the **twisted sum** of a separable normed space and the real line. Recall that a real function F on a normed space E is **quasi-linear** if

(0) (i) $F(rx) = rF(x)$ for all scalars r and all x in E;

(ii) $|F(x+y) - F(x) - F(y)| \le C(\|x\| + \|y\|)$ for all x, y in E and some constant C.

Now define the **twisted sum** of the real line and E (with respect to F) as the vector space $X_F = \mathbb{R} \times E$ equipped with quasi-norm $|||(r,x)||| = |r - F(x)| + \|x\|$. It is easy to verify that

$$|||(r_1 + r_2, x_1 + x_2)||| \le (C+1)[|||(r_1, x_1)||| + |||(r_2, x_2)|||].$$

The space E is said to be a K-**space** if the subspace $\mathbb{R} \times \{0\}$ is complemented in X_F for every quasi-linear map F on E. (This is a slight abuse of terminology; strictly speaking, it is the completion of E that is the K-space.) So we are interested in Klee's question for the non-K spaces. The only known non-K spaces are ℓ_1-like. The **Ribe function** is defined on ℓ_1^0, the space of finitely supported elements of ℓ_1, by

$$F_0(x) = \sum_i x_i \ell\text{n}|x_i| - \left(\sum_i x_i\right) \ell\text{n}\left|\sum_i x_i\right|$$

with the convention that $0\ell\text{n}0 = 0$. Ribe [8] proved that F_0 is quasi-linear on ℓ_1^0 and used F_0 to show that ℓ_1 is not a K-space. Closely related functions were used by Kalton [2] and Roberts [9] to prove the same result. The reflexive space $\ell_2(\ell_1^n)$ is not a K-space, and the B-convex spaces are K-spaces [2]. Kalton and Roberts [4] showed that c_0 and ℓ_∞ are K-spaces. It is not known whether the James space is a K-space. We are studying Klee's problem for spaces E and quasi-linear maps F on E such that $\mathbb{R} \times \{0\}$ is not complemented in X_F. By Theorem 2.5 of [3], there is no linear map T on E such that $|T(x) - F(x)| \le C\|x\|$ for all x in E (i.e. F does not split on E). The corollary to our main theorem implies that none of the spaces above can be a counterexample for Klee's question, since the F concerned does split on an infinite-dimensional subspace.

We now state our main result:

MAIN THEOREM: *Let E be an $\aleph_0$-dimensional normed space. Assume F is a quasi-linear function on E for which there are a linearly independent sequence (x_i) in E and a linear map T on* span(x_i) *such that*

(1) $|T(x) - F(x)| \leq C\|x\|$ *for all x in* span(x_i) *and some constant C.*

Then there are a trivial dual topology τ_2 on $\mathbb{R} \times E$, weaker than the quasi-norm topology, and a τ_2-neighborhood U of 0 such that if $(r, x) \in U$ and $\|x\| \leq 1$, then $|||(r, x)||| < C$ for some constant C.

Before we prove the theorem, we set the framework for the construction with some auxiliary results. We begin with:

Definition: Suppose (G_i) is a finite or infinite sequence of subsets of E, and (n_i) is a sequence of positive integers (of the same length as (G_i)). The (n_i)-**sum** of (G_i) is the set of all finite sums

$$z = r_1 z_1 + r_2 z_2 + r_3 z_3 + \cdots$$

where $|r_i| \leq 1$ for all i and $z_1, \dots, z_{n_1}$ are in G_1, $z_{n_1+1}, \dots, z_{n_1+n_2}$ are in G_2, $z_{n_1+n_2+1}, \dots, z_{n_1+n_2+n_3}$ are in G_3, etc. Note that if $|r| \leq 1$, rz is also in the (n_i)-sum. ∎

LEMMA 1: *Let X be a vector space and let (U_n) be a neighborhood base at 0 for a pseudo-metrizable vector topology on X, chosen so that $U_{n+1} + U_{n+1} \subset U_n$ for all n and $[-1, 1]U_n \subset U_n$ for all n. Let (F_n) be a sequence of subsets of X, chosen so that $[-1, 1]F_n \subset F_n$ and $F_{n+1} + F_{n+1} \subset F_n$, for all n. Then the sequence $(U_n + F_n)$ is a neighborhood base at 0 for a pseudo-metrizable vector topology on X which is weaker than the original topology.*

Proof: Immediate. ∎

In the next lemma, we specify F_n more closely.

LEMMA 2: *Let X and (U_n) be as in Lemma 1. Let (G_n) be a sequence of subsets of X. Define subsets F_n of X as follows: for each n in N, F_n is the (2^{i-n})-sum of the G_i's for $i \geq n$. Then (F_n) satisfies the hypotheses of Lemma 1.*

Proof: $[-1, 1]F_n \subset F_n$ as remarked already. For a typical sum in $F_{n+1} + F_{n+1}$, at most $2 \cdot 2^{i-(n+1)} = 2^{i-n}$ of the z_i's are in G_i for $i \geq n+1$, so $F_{n+1} + F_{n+1} \subset F_n$. ∎

Remark: Note that there is an apriori bound on the number of elements of G_i appearing in a sum in F_n, for any n: the bound is 2^{i-1}; we use the looser bound 2^i. ∎

In our construction, (U_n) is a neighborhood base at 0 for the twisted sum topology. The G_i's of Lemma 2 will be chosen so that (U_n+F_n) is a neighborhood base at 0 for a trivial dual topology τ_2; they will also have to be chosen so that τ is the supremum of τ_1 and τ_2. The next lemma identifies the topology τ_1:

LEMMA 3: *Let F be a quasi-linear map on a normed space E and let $X_F = \mathbb{R} \times E$ with the quasi-norm $|||(r,x)||| = |r - F(x)| + \|x\|$. Assume $\mathbb{R} \times \{0\}$ is not complemented in X_F. Then the strongest nearly convex topology on $\mathbb{R} \times E$ which is weaker than the quasi-norm topology has a neighborhood base at 0 of sets of the form $\{(r,x)\colon \|x\| < \eta\}$.*

Proof: Sets of the above type are a neighborhood base at 0 for a nearly convex topology weaker than τ, the quasi-norm topology. The closure of $\{0\}$ for this weaker topology is $\mathbb{R} \times \{0\}$; and if (r,x) is in X_F and $x \neq 0$, there is f in E^* with $f(x) \neq 0$. Then $f(\pi(r,x)) \neq 0$, where π is the quotient map of X_F onto E.

Now suppose ν is a nearly convex topology on $\mathbb{R} \times E$, weaker than the quasi-norm topology. Since $\mathbb{R} \times \{0\} = K(\tau)$ is not complemented, the ν-closure of $\{0\}$ must contain $\mathbb{R} \times \{0\}$. Let U be ν-open containing 0. Choose V ν-open containing 0, with $V + V \subset U$. Choose $\epsilon > 0$ so that if $|||(r,x)||| < \epsilon$, then $(r,x) \in V$.

Now, $|||(F(x),x)||| = \|x\|$, so if $\|x\| < \epsilon$, then $(F(x),x) \in V$. Also, $(r - F(x), 0)$ is in V since it is in the ν-closure of 0, and so (r,x) is in U. ∎

Notation: Let Z be a Banach space with a basis (v_i). Let (v_i^*) be the coordinate functionals on Z. For n a positive integer and x in Z, set $x\,|_{[1,n]} = \sum_{i=1}^n v_i^*(x)v_i$, and $x\,|_{(n,\infty)} = \sum_{i=n+1}^\infty v_i^*(x)v_i$. Say that x **is to the right of** n if $v_i^*(x) = 0$ for $i \leq n$. ∎

We need two more preliminary results before proving our main theorem:

LEMMA 4: *Let Z be a Banach space with a monotone basis (v_i), let K be a compact subset of Z, and let $\epsilon > 0$. Then there is n so that if y is to the right of n and $x \in K$, then $\|x\| < \|x+y\| + \epsilon$.*

Proof: Choose n so that $\|x\,|_{(n,\infty)}\| < \epsilon$ for every x in K. Now if y is to the right of n, then $\|x\,|_{[1,n]}\| = \|(x+y)\,|_{[1,n]}\| \leq \|(x+y)\|$, since the basis is monotone, so $\|x\| < \|x+y\| + \epsilon$. ∎

LEMMA 5: *Let y_i, $1 \le i \le k$, be linearly independent elements of a normed space E. Define $y_{k+1} = -\sum_{i=1}^{k} y_i$, and let $\eta > 0$. Set $z_i = my_i$, $1 \le i \le k+1$, for some $m > 0$. Then we can choose m so large that the following condition is satisfied: if at most k of the r_i's are non-zero, and if $\|\sum_{i=1}^{k+1} r_i z_i\| < 3$, then $\sum_{i=1}^{k+1} |r_i| < \eta$.*

Proof: Choose $M > 0$ so that $\sum_{i=1}^{k} |\alpha_i| \le M \|\sum_{i=1}^{k} \alpha_i y_i\|$ for all k-tuples (α_i). Now suppose $\|\sum_{i=1}^{k+1} r_i z_i\| < 3$, with at most k r_i's non-zero. If $r_{k+1} = 0$, then

$$\sum_{i=1}^{k} |r_i| \le \frac{3M}{m} < \eta$$

for $m > 3M/\eta$.

If $r_{k+1} \ne 0$, then some other r_i is 0, r_1, say; now,

$$\Big\|\sum_{i=1}^{k+1} r_i z_i\Big\| = \Big\| -mr_{k+1}y_1 + \sum_{i=2}^{k} m(r_i - r_{k+1})y_i\Big\| < 3.$$

Therefore

$$|r_{k+1}| + \sum_{i=2}^{k+1} |r_i - r_{k+1}| < \frac{3M}{m},$$

so

$$\sum_{i=2}^{k+1} |r_i| \le |r_{k+1}| + \sum_{i=2}^{k+1} |r_i - r_{k+1}| + k|r_{k+1}|$$
$$< \frac{3(k+1)M}{m} < \eta$$

for $m > 3(k+1)M/\eta$. ∎

Proof of Main Theorem: We will construct inductively the sets G_n used in Lemma 2. That lemma will give us the sets F_n and then Lemma 1 will provide the topology.

We may assume that for x_i and T in the theorem, $T(x_i) = 0$ for each i. This is possible since for each i, there is a scalar α_i so that $T(x_{2i-1} + \alpha_i x_{2i}) = 0$; now the sequence $x_i' = x_{2i-1} + \alpha_i x_{2i}$ also satisfies condition (1) of the Theorem.

We can regard E as a subspace of the Banach space $Z = C[0,1]$, which has a monotone basis. Any positive scalar multiple of the quasi-norm yields the same topology as the quasi-norm, so we can and do assume that the constant C in 0(ii) above is 1. This can be done by multiplying F by a suitable positive constant.

Finally, we use $\| \quad \|$ to refer to the norm on Z and on E. We only calculate norms of elements of E, but we do use the monotonicity of (v_i) in Z.

Now we begin the construction of (G_i).

Choose $0 < c_n \le 2^{-(n+3)}$ (and thus $\sum_{n=1}^{\infty} c_n < \frac{1}{4}$). Let (d_j) be any sequence whose linear span is E, and let (e_i) be an indexing of (d_j) such that each d_j occurs infinitely often in (e_i). We can assume that $\|d_j\| \le 1$ and that $|F(d_j)| \le 1$ for each j, by multiplying d_j by a positive constant.

Assume that finite sets $G_0, G_1, \ldots, G_{n-1}$ have been constructed, with $G_0 = \{0\}$, satisfying the following conditions:

(2) for each $1 \le i \le n-1$, G_i is a finite set $(w_{i,j} : 1 \le j \le 2^{i+1})$, with

$$w_{i,j} = e_i + m_i x_{\ell(i,j)}, \quad j \le 2^i,$$

$$w_{i,2^i+1} = e_i - \sum_{j=1}^{2^i} m_i x_{\ell(i,j)};$$

here, (x_i) is the sequence in the statement of the theorem.

(3) Set $z_{i,j} = w_{i,j} - e_i$. Then if $\| \sum_{j=1}^{2^i+1} r_j z_{i,j} \| < 3$ with at most 2^i r_j's non-zero, $\sum_{j=1}^{2^i+1} |r_j| < c_i$.

To define G_n, let K_n' be the (2^i)-sum of G_i for $i \le n-1$, (so $K_1' = \{0\}$) and let $K_n = K_n' + [-2^n, 2^n]e_n$. Then K_n is a compact subset of $E \subset Z$. By Lemma 4, there is an integer s_n such that if y is to the right of s_n then $\|x\| < \|x+y\| + c_n$ for all x in K_n. By the linear independence of the sequence (x_i), we can choose $x_{\ell(n,1)}, \ldots, x_{\ell(n,2^n)}$, all to the right of s_n, with $\ell(n,j) < \ell(n,j')$ if $j < j'$. For ease of notation, put $x_{n,i} = x_{\ell(n,i)}$, $1 \le i \le 2^n$, and put $x_{n,2^n+1} = -\sum_{i=1}^{2^n} x_{n,i}$.

By Lemma 5, we can choose m_n so large that if

$$\Big\| \sum_{j=1}^{2^n+1} r_{n,j} m_n x_{n,j} \Big\| < 3,$$

with at most 2^n of the $r_{n,j}$ non-zero, then

$$\sum_{j=1}^{2^n+1} |r_{n,j}| < c_n. \tag{4}$$

Finally, for $1 \le i \le 2^n + 1$, put

$$w_{n,i} = e_n + m_n x_{n,i}$$

and let

$$G_n = (w_{n,i}), \quad 1 \le i \le 2^n + 1.$$

Note that since $\sum_{i=1}^{2^n+1} x_{n,i} = 0$, $e_n \in \mathrm{co}G_n$. (We denote the convex hull of A by $\mathrm{co}A$.) This finishes the construction of (G_i).

Now let (F_n) be the subsets of E used in Lemma 1: F_n is the (2^{i-n}) sum of (G_i) for $i \ge n$. Let (U_n) be a neighborhood base at 0 for the quasi-norm topology on $\mathbb{R} \times E$, with $U_{n+1} + U_{n+1} \subset U_n$ and $[-1,1]U_n \subset U_n$, for all n; also assume that $|||w||| < 1$ if $w \in U_1$. Let τ_2 be the topology yielded by Lemma 1.

We claim that τ_2 is trivial dual. To see this, note that for $m \ge n$, $e_m \in \mathrm{co}(w_{m,i}) \subset \mathrm{co}F_n \subset \mathrm{co}(F_n + U_n)$; since each d_j occurs infinitely often in the sequence (e_m), $K(\tau_2)$ contains every d_j and therefore contains $\{0\} \times E$. Also, $(1,0) \in \mathrm{co}U_n \subset \mathrm{co}(U_n + F_n)$ for every n, so $K(\tau_2)$ contains $\mathbb{R} \times \{0\}$. This proves the claim.

Now suppose that

$$x = \sum_{i=1}^{n} \sum_{j=1}^{2^i+1} r_{i,j}(e_i + m_i x_{i,j})$$

is in F_1 and that $\|x\| < 1$. We will first prove that $|F(x)| < 9$.

Toward that end: since the $x_{n,j}$ are to the right of s_n, the construction of G_n implies that

$$\Big\| \sum_{i=1}^{n-1} \sum_{j=1}^{2^i+1} r_{i,j}(e_i + m_i x_{i,j}) + \sum_{j=1}^{2^n+1} r_{n,j} e_n \Big\| < 1 + c_n \tag{5}$$

from which, since $\|x\| < 1$,

$$\Big\| \sum_{j=1}^{2^n+1} r_{n,j} m_n x_{n,j} \Big\| < 2 + c_n < 3. \tag{6}$$

Now from (4) and (6), we have

$$\sum_{j=1}^{2^n+1} |r_{n,j}| < c_n; \tag{7}$$

combining this with (5), we have

$$\Big\| \sum_{i=1}^{n-1} \sum_{j=1}^{2^i+1} r_{i,j}(e_i + m_i x_{i,j}) \Big\| < 1 + 2c_n. \tag{8}$$

For the induction step, assume that for some ℓ,

$$\Big\|\sum_{i=1}^{\ell}\sum_{j=1}^{2^i+1} r_{i,j}(e_i+m_ix_{i,j})\Big\| < 1+2c_n+\cdots+2c_{\ell+1}. \tag{9}$$

Since the $x_{\ell,j}$ are to the right of s_ℓ, the construction of G_ℓ implies that

$$\Big\|\sum_{i=1}^{\ell-1}\sum_{j=1}^{2^i+1} r_{i,j}(e_i+m_ix_{i,j}) + \sum_{j=1}^{2^\ell+1} r_{\ell,j}e_\ell\Big\| < 1+2c_n+\cdots+2c_{\ell+1}+c_\ell, \tag{10}$$

from which

$$\Big\|\sum_{j=1}^{2^\ell+1} r_{\ell,j}m_\ell x_{\ell,j}\Big\| < 2+4c_n+\cdots+4c_{\ell+1}+c_\ell < 3. \tag{11}$$

Now from (4) and (11), we have

$$\sum_{j=1}^{2^\ell+1} |r_{\ell,j}| < c_\ell; \tag{12}$$

combining this with (10), we obtain

$$\Big\|\sum_{i=1}^{\ell-1}\sum_{j=1}^{2^i+1} r_{i,j}(e_i+m_ix_{i,j})\Big\| < 1+2c_n+\cdots+2c_\ell, \tag{13}$$

recalling that $\|e_i\| \le 1$. This finishes the induction step.

The above argument has yielded that

$$\Big\|\sum_{j=1}^{2^i+1} r_{i,j}e_i\Big\| < c_i \tag{14}$$

for each i; from this and $\|x\| < 1$, we have

$$\Big\|\sum_{i=1}^{n}\sum_{j=1}^{2^i+1} r_{i,j}m_ix_{i,j}\Big\| < 1+\sum_{n=1}^{\infty} c_n < 2. \tag{15}$$

From (15) and (1), recalling $T(x_{i,j})=0$,

$$\left|F\left(\sum_{i=1}^{n}\sum_{j=1}^{2^i+1} r_{i,j}m_ix_{i,j}\right)\right| < 2.$$

To estimate

$$F\left(\sum_{i=1}^{n}\sum_{j=1}^{2^i+1} r_{i,j}e_i\right),$$

recall that $|F(e_i)| \le 1$ for each i, so

$$|F\left(\sum_{j=1}^{2^i+1} r_{i,j}e_i\right)| < 2^{-i}.$$

Therefore

$$\begin{aligned}|F\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{2^i+1} r_{i,j}e_i\right)\right)| &\le \sum_{i=1}^{n} 2^{-i} + \sum_{i=1}^{n} i\|\sum_{j=1}^{2^i+1} r_{i,j}e_i\| \\ &< 1 + \sum_{i=1}^{n} i\cdot 2^{-i} < 4\end{aligned}$$

(using $\left|F(\sum u_i)\right| \le \sum |F(u_i)| + \sum i\|u_i\|$). Finally,

$$\begin{aligned}|F(x)| &\le 2+4+\|\sum_{i=1}^{n}\sum_{j=1}^{2^i+1} r_{i,j}e_i\| + \|\sum_{i=1}^{n}\sum_{j=1}^{2^i+1} r_{i,j}m_i x_{i,j}\| \\ &< 2+4+1+2 = 9.\end{aligned}$$

To complete the proof of the theorem, suppose $(r,x) \in U_1 + F_1$ and $\|x\| \le 1$. Write $(r,x) = (r,y)+(0,z)$, with $(r,y) \in U_1$ and $z \in F_1$. Then $|r-F(y)|+\|y\| \le 1$; from this and $\|x\| \le 1$ follows $\|z\| \le 2$. Now since $z \in F_1$, the preceding paragraph implies $|F(z)| < 18$. At last,

$$\begin{aligned}|r - F(x)| &\le |r - F(y)| + |F(y) - F(x)| \\ &\le 1 + |F(y) - F(x)| \\ &\le 1 + |F(z)| + \|z\| + \|x\| \\ &< 22,\end{aligned}$$

so $|||(r,x)||| < 23$. The proof is complete. ∎

COROLLARY: *Let E be a separable normed space and let E_0 be an $\aleph_0$-dimensional subspace of E which is dense in E. Assume that there is a quasi-linear map F on E_0 which splits on an infinite-dimensional subspace of E_0. Then the twisted* *sum topology on* $\mathbb{R} \otimes_F E$ *is the supremum of a trivial dual topology and a nearly exotic topology.*

Proof: Let q denote the quotient map of $\mathbb{R} \otimes_F \tilde{E}$ onto $\tilde{E}$, where $\tilde{E}$ is the completion of E. (For $x \in E_0, q(r,x) = x$.) The subspace E_0 satisfies the hypotheses of the main theorem. Therefore there are a trivial dual topology τ_2 on $\mathbb{R} \times E_0$, weaker than the twisted sum topology; a τ_2-neighborhood V of 0; and a constant C so that if $x \in E_0, (r,x) \in V$, and $\|x\| < 1$, then $|||(r,x)||| < C$.

We can assume that V contains a τ_2-neighborhood U of 0 of the form $B_\alpha + F_n$, where $F_n \subset E_0$ is as constructed as in the proof of the main theorem, and for any $\beta > 0$,

$$B_\beta = \{(r,x) \in \mathbb{R} \times E_0 : |||(r,x)||| < \beta\}.$$

Sets of the form $\overline{B_\beta} + q^{-1}(F_m)$, where

$$\overline{B_\beta} = \{w \in \mathbb{R} \otimes_F E : |||w||| < \beta\},$$

obviously form a neighborhood base at the origin for a vector topology $\overline{\tau}_2$ on $\mathbb{R} \otimes_F E$, weaker than the twisted sum topology. The topology $\overline{\tau}_2$ is trivial dual since its restriction to the dense subspace $\mathbb{R} \times E_0$ is trivial dual.

Now choose $0 < \gamma < 1/2$ so that $\overline{B_\gamma} + \overline{B_\gamma} \subset \overline{B_\alpha}$, and assume that $w \in \overline{B_\gamma} + q^{-1}(F_n)$ and $\|q(w)\| < 1/2$. Choose $w_0 \in \mathbb{R} \times E_0$ so that $|||w - w_0||| < \gamma$. Then $\|q(w) - q(w_0)\| < \gamma$, so $\|q(w_0)\| < \gamma + 1/2 < 1$. Clearly, $w_0 \in \overline{B_\alpha} + q^{-1}(F_n)$, and so from our assumption, $|||w_0||| < C$. Now, $|||w||| < (\alpha/\gamma)(C+1)$, and the proof is complete. ∎

The theorem and corollary apply to several spaces which are either not K-spaces or for which it is not known whether they are K-spaces:

THEOREM: *For the following pairs of normed spaces E and quasi-linear maps F on E, the twisted sum topology on $X_F = \mathbb{R} \times E$ is the supremum of a nearly convex topology and a trivial dual topology:*

(a) *E is any infinite-dimensional subspace of ℓ_1^0 (whether or not it is a K-space), F is the Ribe function F_0;*
(b) *E is the linear span of the usual unit vector basis for the James space, under the James norm; F is any quasi-linear function on E;*
(c) *E is the span of the usual unit vector basis in $\ell_p(\ell_1^n)$, for $1 < p < \infty$ (this is a reflexive non-K space); F will be described below.*

Proof: For (a), let $H = \{x \in E\colon \sum_i x_i = 0\}$. Note that if x, $y \in H$ and x and y have disjoint supports, $F_0(x+y) = F_0(x) + F_0(y)$. Since H has codimension at most 1 in E and E is infinite dimensional, there is a sequence of non-zero elements (x_i) in H satisfying sup (support x_i) < inf (support x_{i+1}) for all i.

As remarked above, F_0 is linear on span(x_i), so if we define $T(x_i) = F(x_i)$, the linear function T certainly satisfies hypothesis (1) of the theorem. Therefore the theorem applies to E.

For (b), it is known that the even unit vectors e_{2n} span a pre-Hilbert subspace of the James space (see [1]). The B-convexity of span(e_{2n}) and Theorems 2.6 of [2] and 2.5 of [3] imply that there is a linear map T on span(e_{2n}) such that $|T(x) - F(x)| \leq C\|x\|$ for all x in span(e_{2n}). Therefore the theorem applies.

(c) For each n let $(e_{i,n})$ be the usual unit vector basis of ℓ_1^n, and let E be the span of the $e_{i,n}$ in $\ell_p(\ell_1^n)$. Let (c_n) be any sequence in ℓ_q, $\frac{1}{p} + \frac{1}{q} = 1$. Let F_0 be the Ribe function and define F on E by

$$F((x_n)) = \sum_n c_n F_0(x_n).$$

We claim that F is quasi-linear. For this, if (x_n) and (y_n) are in E, the sequences $(\|x_n\|_1)$ and $(\|y_n\|_1)$ are ℓ_p sequences, and for each n,

$$|c_n F_0(x_n + y_n) - c_n F_0(x_n) - c_n F_0(y_n)| \leq c_n(\|x_n\|_1 + \|y_n\|_1).$$

From Hölder's inequality,

$$|F((x_n + y_n)) - F((x_n)) - F((y_n))| \leq \|(c_n)\|_q(\|(x_n)\|_p + \|(x_n)\|_p).$$

Theorem 4.7 of [2] gives that E is not a K-space. The F just defined proves this directly, for suppose there is a linear T on E with $|T(x) - F(x)| \leq C\|x\|$ for all x in E. Then since $F(e_{i,n}) = 0$ for all i, n, $|T(e_{i,n})| \leq C$ for all i, n. But

$$F\left(\frac{1}{n}\sum_{i=1}^{n} e_{i,n}\right) = -c_n \log n,$$

a contradiction if we choose c_n so that $(c_n \log n)$ is unbounded.

Finally, our theorem applies in this situation. To show this, for each n pick a unit vector x_n in ℓ_1^n. The sequence (x_n) is equivalent to the usual basis of ℓ_p, which is B-convex; the results already mentioned imply that there is a linear T

on span(x_n) such that $|T(x) - F(x)| \leq C\|x\|$ for all x in span(x_n). This finishes the proof. ∎

Note that, because of the separability, the corollary applies to the completions of the twisted sums in (a)–(c) above.

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